Tensor Products of p-adic Vector Measures

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Abstract

Tensor products of p-adic vector measures are introduced and some of their properties are investigated. It is shown that a Fubini's Theorem holds for tensor products of τ-additive vector measures.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [14] or [15]). For E a locally convex space, we will denote by cs(E) the collection of all continuous seminorms on E. If E is Hausdorff, $\hat{E}$ denotes the completion of E. If F is another locally convex space, then, for $p \in cs(E)$ and $q \in cs(F)$, $p \otimes q$ denotes the tensor product of the seminorms p, q. On the tensor product $E \otimes F$ we will consider the projective topology, which coincides with the topology generated by the seminorms $p \otimes q$, $p \in cs(E)$ and $q \in cs(F)$. For a zero-dimensional Hausdorff space X, $\beta_o X$ is the Banachewski compactification of X (see [4]), $C(X, E)$ is the space of all continuous E-valued functions on X, while $C_b(X, E)$ and $C_{rc}(X, E)$ are the subspaces of all $f \in C(X, E)$ whose range is bounded or relatively compact in E, respectively. In case $E = \mathbb{K}$, we write simply $C(X)$, $C_b(X)$ and $C_{rc}(X)$, respectively. For $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\|_A = \sup\{|f(x)| : x \in A\} \quad \text{and} \quad \|f\| = \|f\|_X.$$

For $A \subset X$, A^c will be its complement in X and χ_A the $\mathbb{K}$-valued characteristic function of A. Next we will recall the definition of the topologies β and β_o on $C_b(X)$ (see [5] and [6]). Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $Z \in \Omega$, let C_Z be the set of all $h \in C_{rc}(X)$ for which the continuous extension h^{β_o} to all of $\beta_o X$ vanishes on Z. We denote by β_Z the locally convex topology on $C_b(X)$ generated by the seminorms p_h, $h \in C_Z$,

$p_h(f) = \|hf\|$.The inductive limit of the topologies β_Z, $Z \in \Omega$, is the topology β (see [5]). As it is shown in [8], Theorem 2.2, an absolutely convex subset W of $C_b(X)$ is a β_Z neighborhood of zero iff, for each $r > 0$, there exist a clopen (i.e. both closed and open) subset A of X, whose closure in $\beta_o(X)$ is disjoint from Z, and $\epsilon > 0$ such that

$$\{f \in C_b(X) : \|f\|_A \leq \epsilon, \|f\| \leq r\} \subset W.$$

The strict topology β_o (see [5]) is defined by the seminorms $f \mapsto \|\phi f\|$, where ϕ ranges over the family $B_{ou}(X)$ of all $\phi \in \mathbb{K}^X$ which are bounded and vanish at infinity. As it is shown in [11], Theorem 4.18, the topologies β and β_o on $C_b(X)$ coincide.

Assume next that X is a non-empty set and $\mathcal{R}$ a separating algebra of subsets of X, i.e. $\mathcal{R}$ is a family of subsets of X such that

1. $X \in \mathcal{R}$, and, if $A, B \in \mathcal{R}$, then $A \cup B$, $A \cap B$, A^c are also in $\mathcal{R}$.
2. If x, y are distinct elements of X, then there exists a member of $\mathcal{R}$ which contains x but not y.

Then $\mathcal{R}$ is a base for a Hausdorff zero-dimensional topology $\tau_{\mathcal{R}}$ on X. For E a locally convex space, we denote by $M(\mathcal{R}, E)$ the space of all finitely-additive measures $m : \mathcal{R} \to E$ such that $m(\mathcal{R})$ is a bounded subset of E (see [11]). For a net (V_δ) of subsets of X, we write $V_\delta \downarrow \emptyset$ if (V_δ) is decreasing and $\cap V_\delta = \emptyset$. An element m of $M(\mathcal{R}, E)$ is said to be τ-additive if $m(V_\delta) \to 0$ for each net (V_δ) in $\mathcal{R}$ with $V_\delta \downarrow \emptyset$. We will denote by $M_\tau(\mathcal{R}, E)$ the space of all τ-additive members of $M(\mathcal{R}, E)$. For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p : \mathcal{R} \to \mathbb{R}, \quad m_p(A) = \sup\{p(m(V)) : V \in \mathcal{R}, V \subset A\} \quad \text{and} \quad \|m\|_{p|} = m_p(X).$$

We also define

$$N_{m,p} : X \to \mathbb{R}, \quad N_{m,p}(x) = \inf\{m_p(V) : x \in V \in \mathcal{R}\}.$$

Next we will recall the definition of the integral of an $f \in \mathbb{K}^X$ with respect to some $m \in M(\mathcal{R}, E)$. Assume that E is a complete Hausdorff locally convex space. For $A \subset X$, let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, A_2, \dots, A_n; x_1, x_2, \dots, x_n\}$, where $\{A_1, A_2, \dots, A_n\}$ is an $\mathcal{R}$-partition of A and $x_k \in A_k$. We make $\mathcal{D}_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ if the partition of A in α_1 is a refinement of the one in α_2. For $\alpha = \{A_1, A_2, \dots, A_n; x_1, x_2, \dots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{k=1}^n f(x_k)m(A_k)$. If the limit $\lim \omega_\alpha(f, m)$ exists in E, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$ (see [11]). For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in \mathcal{R}$ and $\int_A f\, dm = \int \chi_A f\, dm$. If f is bounded on A, then

$$p\left(\int_A f\, dm\right) \leq \|f\|_A \cdot m_p(A).$$

Assume next hat m is τ-additive and let $S(\mathcal{R})$ be the space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions. Let G_m be the space of all $f \in \mathbb{K}^X$ for which

$$\|f\|_{N_{m,p}} = \sup_{x \in X} |f(x)| \cdot N_{m,p}(x) < \infty$$

for all $p \in cs(E)$. We consider on G_m the locally convex topology generated by the seminorms $\|.\|_{N_{m,p}}$, $p \in cs(E)$. The map

$$\omega : S(\mathcal{R}) \to E, \quad \omega(g) = \int g\, dm$$

is continuous and so it has a continuous extension $\bar{\omega} : \overline{S(\mathcal{R})} \to E$. We will say that f is (VR)-integrable with respect to m iff $f \in \overline{S(\mathcal{R}} = L_m$. In this case we will denote $\bar{\omega}(f)$ by $(VR)\int f\, dm$ (see [11]). As it is shown in [11], the space L_m, with the induced topology, is complete. In the same paper it is proved that, if $f \in \mathbb{K}^X$ is m-integrable, then it is also (VR)-integrable and

$$(VR)\int f\, dm = \int f\, dm.$$

2 Tensor Products of Measures

Let $\mathcal{R}_1$, $\mathcal{R}_2$ be separating algebras of subsets of the sets X, Y, respectively, and let $\mathcal{R}$ be the algebra of subsets of $X \times Y$ which is generated by the family

$$\mathcal{R}_1 \times \mathcal{R}_2 = \{A \times B : A \in \mathcal{R}_1, B \in \mathcal{R}_2\}.$$

Lemma 2.1 *If $\mathcal{R}$, $\mathcal{R}_1$, and $\mathcal{R}_2$ are as above, then :*

1. *A subset G of $X \times Y$ is in $\mathcal{R}$ iff it is a finite union of members of $\mathcal{R}_1 \times \mathcal{R}_2$.*
2. *Every member of $\mathcal{R}$ is a finite union of pairwise disjoint members of $\mathcal{R}_1 \times \mathcal{R}_2$.*

Proof : (1). Let Φ be the family of all finite unions of members of $\mathcal{R}_1 \times \mathcal{R}_2$. It is clear that Φ contains $X \times Y$ and that it is closed under finite intersections and finite unions. Also, since $(A \times B)^c = [A^c \times Y] \bigcup [A \times B^c]$, it follows easily that Φ is closed under complimentation. It is clear now that $\Phi = \mathcal{R}$.
(2). Let $G = \bigcup_{k=1}^n A_k \times B_k$, where $A_k \in \mathcal{R}_1$, $B_k \in \mathcal{R}_1$. We will show by induction on n that G is a finite union of pairwise disjoint members of $R_1 \times \mathcal{R}_2$. Suppose that it is true when $n = k$ and let $n = k + 1$. By our induction hypothesis, there are pairwise disjoint members $D_i \times F_i$ of $R_1 \times \mathcal{R}_2$. $i = 1, \dots, N$, such that

$$\bigcup_{i=1}^{N} D_i \times F_i = \bigcup_{i=1}^{k} A_i \times B_i.$$

Let

$$\Phi_i = \{D_i^c \times Y, D_i \times F_i^c\} \quad \text{for} \quad i = 1, \dots, N,$$

and let $\mathcal{F}$ be the family of all subsets of $X \times Y$ of the form $\bigcap_{i=1}^N V_i$, where $V_i \in \Phi_i$. Clearly $\bigcup \mathcal{F} = \left(\bigcup_1^k A_i \times B_i\right)^c$ and the members of $\mathcal{F}$ are pairwise disjoint. Now

$$\bigcup_{i=1}^{k+1} A_i \times B_1 = \left[\bigcup_1^N D_i \times F_i\right] \bigcup \{D \cap (A_{k+1} \times B_{k+1}) : D \in \mathcal{F}\}.$$

This clearly shows that $\bigcup_1^{k+1} A_i \times B_i$ can be written as a finite union of pairwise disjoint members of $R_1 \times \mathcal{R}_2$ and the Lemma follows.

Lemma 2.2 $\tau_{\mathcal{R}} = \tau_{\mathcal{R}_1} \times \tau_{\mathcal{R}_2}$.

Proof : Since $\tau_o = \tau_{\mathcal{R}_1} \times \tau_{\mathcal{R}_2}$ is zero dimensional and $\mathcal{R}_1 \times \mathcal{R}_2 \subset \tau_o$, it follows that $\tau_{\mathcal{R}} \subset \tau_o$. On the other hand, let $(x, y) \in G \in \tau_o$. There are $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$ with $(x, y) \in A \times B \subset G$. Since $A \times B \in \mathcal{R}$, it follows that $\mathcal{R}$ is a base for τ_o and so $\tau_o = \tau_{\mathcal{R}}$.

Lemma 2.3 *Let E, F be Hausdorff locally convex spaces and $m_1 \in M(\mathcal{R}_1, E)$, $m_2 \in M(\mathcal{R}_2, F)$. If $\{A, A_1, \dots, A_n\} \subset \mathcal{R}_1$ and $\{B, B_1, \dots, B_n\} \subset \mathcal{R}_2$ are such that the sets $A_k \times B_k$, $k = 1, \dots, n$, are pairwise disjoint and their union is $A \times B$, then*

$$m_1(A) \otimes m_2(B) = \sum_{k=1}^{n} m_1(A_k) \otimes m_2(B_k).$$

Proof : We will prove it by induction on n. Suppose that it holds for $n = k$ and let $n = k + 1$. If one of the $A_i \times B$ is empty, we are done. Assume that none of them is empty. Then $A \cap A_i = A_i$ and $B \cap B_i = B_i$. Now

$$\begin{aligned}\bigcup_{i=1}^{k} A_i \times B_i &= (A \times B) \bigcap [(A_{k+1} \times B_{k+1})^c] \\ &= (A \times B) \bigcap \left[(A_{k+1}^c \times Y) \bigcup (A_{k+1} \times B_{k+1}^c)\right].\end{aligned}$$

Also

$$\begin{aligned}(A \cap A_{k+1}^c) \times B &= \left[(A \cap A_{k+1}^c) \times B\right] \bigcap \left[\bigcup_{i=1}^{k} A_i \times B_i\right] \\ &= \bigcup_{i=1}^{k} \left(A_i \cap A_{k+1}^c\right) \times B_i\end{aligned}$$

and

$$\begin{aligned}A_{k+1} \times (B \cap B_{k+1}^c) &= \left\{A_{k+1} \times (B \cap B_{k+1}^c)\right\} \bigcap \left\{\bigcup_{i=1}^{k} A_i \times B_i\right\} \\ &= \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) \times (B_i \cap B_{k+1}^c).\end{aligned}$$

By our induction hypothesis, we have

$$m_1(A \cap A_{k+1}^c) \otimes m_2(B) = \sum_{i=1}^{k} m_1(A_i \cap A_{k+1}) \otimes m_2(B_i)$$

and

$$m_1(A_{k+1}) \otimes m_2(B \cap B_{k+1}^c) = \sum_{i=1}^{k} m_1(A_i \cap A_{k+1}) \otimes m_2(B_i \cap B_{k+1}^c).$$

Moreover, for $i \le k$,

$$\begin{aligned} m_1(A_i) \otimes m_2(B_i) &= m_1(A_i \cap A_{k+1}) \otimes m_2(B_i) + m_1\left(A_i \cap A_{k+1}^c\right) \otimes m_2(B_i) \\ &= m_1(A_i \cap A_{k+1}) \otimes m_2(B_i \cap B_{k+1}) \\ &+ m_1(A_i \cap A_{k+1}) \otimes m_2(B_i \cap B_{k+1}^c) + m_1(A_i \cap A_{k+1}^c) \otimes m_2(B_i). \end{aligned}$$

Since one of the two sets $A_i \cap A_{k+1}$, $B_i \cap B_{k+1}$ must be empty, we have that

$$m_1(A_i) \otimes m_2 B_i) = m_1(A_i \cap A_{k+1}) \otimes m_2(B_i \cap B_{k+1}^c) + m_1(A_i \cap A_{k+1}^c) \otimes m_2(B_i).$$

Thus

$$\begin{aligned} m_1(A) \otimes m_2(B) &= m_1(A \cap A_{k+1}^c) \otimes m_2(B) + m_1(A_{k+1}) \otimes m_2(B) \\ &= m_1(A \cap A_{k+1}^c) \otimes m_2(B) + m_1(A_{k+1}) \otimes m_2(B \cap B_{k+1}^c) \\ &+ m_1(A_{k+1}) \otimes m_2(B_{k+1}) \\ &= \sum_{i=1}^{k} m_1(A_i) \otimes m_2(B_i) + m_1(A_{k+1}) \otimes m_2(B_{k+1}). \end{aligned}$$

This clearly completes the proof.

Lemma 2.4 *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}$, m_1, m_2 be as in the preceding Lemma. If $A_k, F_i \in \mathcal{R}_1$, $B_k, G_i \in \mathcal{R}_2$, $k = 1, \ldots, n$, $i = 1, \ldots, N$, are such that $\bigcup_{k=1}^n A_k \otimes B_k = \bigcup_{i=1}^N F_i \otimes G_i$ and each of the families $\{A_k \times B_k : k = 1, \ldots, n\}$ and $\{F_i \times G_i : i = 1, \ldots, N\}$ consists of sets which are pairwise disjoint, then*

$$\sum_{k=1}^{n} m_1(A_k) \otimes m_2(B_k) = \sum_{i=1}^{N} m_1(F_i) \otimes m_2(G_i).$$

Proof :

$$A_k \times B_k = \bigcup_{i=1}^{N} (F_i \cap A_k) \times (G_i \cap B_k) \quad \text{and} \quad F_i \times G_i = \bigcup_{k=1}^{n} (F_i \cap A_k) \times (G_i \cap B_k).$$

Now the result follows by applying the preceding Lemma.

Theorem 2.5 *If $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}$, m_1, m_2 are as in the preceding Lemma, then there exists a unique $m \in M(\mathcal{R}, E \otimes F)$ with $m(A \times B) = m_1(A) \otimes m_2(B)$, when $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$.*

Proof : For $G \in \mathcal{R}$, there are $A_i \in \mathcal{R}_1$, $B_i \in \mathcal{R}_2$, $i = 1, \ldots, n$, such that $G = \bigcup_{i=1}^n A_i \times B_i$ and the sets $A_i \times B_i$ are pairwise disjoint. Define $m(G) = \sum_1^n m_1(A_i \otimes m_2(B_i$. In view of the preceding Lemma, m is well defined and finitely additive. Also $m(\mathcal{R})$ is a bounded subset of $E \otimes F$ and so $m \in M(\mathcal{R}, E \otimes F)$. Clearly m is the unique $\mu \in M(\mathcal{R}, E \otimes F)$ such that $\mu(A \times B) = m_1(A) \otimes m_2(B)$ when $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$.

We will call $m_1 \otimes m_2$ the tensor product of m_1, m_2.

Theorem 2.6 *Let $m_1 \in M(\mathcal{R}_1, E)$, $m_2 \in M(\mathcal{R}_2), F)$, $m = m_1 \otimes m_2$, $p \in cs(E)$ and $q \in cs(F)$. Then, for $V_1 \in \mathcal{R}_1$, $V_2 \in \mathcal{R}_2$, we have*

$$m_{p\otimes q}(V_1 \times V_2) = (m_1)_p(V_1) \cdot (m_2)_q(V_2).$$

Moreover, for $x \in X$, $y \in Y$, we have

$$N_{m,p\otimes q}(x,y) = N_{m_1,p}(x) \cdot N_{m_2,q}(y).$$

Proof : Let $d = (m_1)_p(V_1) \cdot (m_2)_q(V_2)$. It is clear that $d \leq m_{p\otimes q}(V_1 \times V_2)$. On the other hand, let $G \in \mathcal{R}$, $G \subset V_1 \times V_2$. There are pairwise disjoint $A_i \times B_i$ in $\mathcal{R}_1 \times \mathcal{R}_2$ such that $G = \bigcup_1^n A_i \times B_i$, $A_i \subset V_1$, $B_i \subset V_2$. Then

$$\begin{aligned} p \otimes q(m(G)) &= p \otimes q\left(\sum_1^n m_1(A_i) \otimes m_2(B_i)\right) \\ &\leq (m_1)_p(V_1) \cdot (m_2)_q(V_2). \end{aligned}$$

Thus $m_{p\otimes q}(V_1 \times V_2) \leq d$ and so $d = m_{p\otimes q}(V_1 \times V_2)$. Given $\epsilon > 0$, there exist $A \in \mathcal{R}_1$ containing x and $B \in \mathcal{R}_2$ containing y such that

$$(m_1)_p(A) < N_{m_1,p}(x) + \epsilon, \quad (m_2)_q(B) < N_{m_2,q}(y) + \epsilon.$$

Thus

$$\begin{aligned} N_{m,p\otimes q}(x,y) &\leq m_{p\otimes q}(A \times B) = (m_1)_p(A) \cdot (m_2)_q(B) \\ &< [N_{m_1,p}(x) + \epsilon] \cdot [N_{m_2,q}(y) + \epsilon]. \end{aligned}$$

Taking $\epsilon \to 0$, we get that

$$N_{m,p\otimes q}(x,y) \leq N_{m_1,p}(x) \cdot N_{m_2,q}(y).$$

On the other hand, let $N_{m,p\otimes q}(x,y) < \theta$. There exists $G \in \mathcal{R}$ containing (x,y) such that $m_{p\otimes q}(G) < \theta$. By Lemma 2.1, there exist $A \in \mathcal{R}_1$ containing x and $B \in \mathcal{R}_2$ containing y such that $A \times B \subset G$. Now

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq (m_1)_p(A) \cdot (m_2)_q(B) = m_{p\otimes q}(A \times B) \leq m_{p\otimes q}(G) < \theta.$$

It is clear now that

$$N_{m,p\otimes q}(x,y) \geq N_{m_1,p}(x) \cdot N_{m_2,q}(y)$$

and the result follows.

Throughout the rest of the paper, E, F will be complete Hausdorff locally convex spaces, X, Y non-empty sets, $\mathcal{R}_1$, $\mathcal{R}_2$ separating algebras of subsets of X, Y, respectively, and $\mathcal{R}$ the algebra of subsets of $X \times Y$ generated by $\mathcal{R}_1 \times \mathcal{R}_2$.

For $f \in \mathbb{K}^X$, $g \in \mathbb{K}^Y$, we will denote by $f \odot g$ the function which is defined on $X \times Y$ by $f \odot g(x,y) = f(x)g(y)$.

Theorem 2.7 *Let $m_1 \in M(\mathcal{R}_1, E)$, $m_2 \in M(\mathcal{R}_2, F)$ and $m = m_1 \otimes m_2$. If $f \in \mathbb{K}^X$ is m_1-integrable and $g \in \mathbb{K}^Y$ is m_2-integrable, then $f \odot g$ is m-integrable and*

$$\int f \odot g\, dm = \left[\int f\, dm_1\right] \otimes \left[\int g\, dm_2\right].$$

Proof : Let $p \in cs(E)$, $q \in cs(F)$ and $\epsilon > 0$. By [11, Theorem 4.2], there are $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$ such that $(m_1)_p(A^c) = (m_2)_q(B^c) = 0$ and f, g are bounded on A, B, respectively. Let $d > \max\{\|f\|_A, \|g\|_B\}$ and choose $0 < \epsilon_1 < \min\{1, \epsilon\}$ such that $d\epsilon_1 \cdot \max\{\|m_1\|_p, \|m_2\|_q\} \leq \epsilon$. By [11, Theorem 4.1], there exist an $\mathcal{R}_1$-partition $\{A_1, \ldots, A_n\}$ of X, which is a refinement of $\{A, A^c\}$, and an $\mathcal{R}_2$-partition $\{B_1, \ldots, B_N\}$ of Y, which is a refinement of $\{B, B^c\}$, such that $|f(x_1) - f(x_2| \cdot (m_1)_p(A_i) < \epsilon_1$, if x_1, x_2 are in A_i, and $|g(y_1) - g(y_2)| \cdot (m_2)_q(B_j) < \epsilon_1$, if y_1, y_2 are in B_j. Choose $x_i \in A_i$, $y_j \in B_j$. Then , by [11, Theorem 4,1], we have

$$p\left(\int f\, dm_1 - \sum_{i=1}^{n} f(x_i) m_1(A_i)\right) \leq \epsilon_1$$

and

$$q\left(\int g\, dm_2 - \sum_{j=1}^{N} g(y_j) m_2(B_j)\right) \leq \epsilon_1$$

We may assume that $\bigcup_{i=1}^{k} A_i = A$ and $\bigcup_{j=1}^{r} B_j = B$. Let $1 \leq i \leq n$, $1 \leq j \leq N$ and let $(z_1, z_2) \in A_i \times B_j$. If either $i > k$ or $j > r$, then $m_{p\otimes q}(A_i \times B_j) = 0$. Suppose that $i \leq k$ and $j \leq r$. Then
$|f(x_i)g(y_j) - f(z_1)g(z_2)| \cdot m_{p\otimes q}(A_i \times B_j)$
$\leq d{\cdot}\max\{|f(x_i - f(z_1)| \cdot (m_1)_p(A_i)\cdot)(m_2)_q(Y), |g(y_j) - g(z_2)| \cdot (m_2)_q(B_j) \cdot (m_1)_p(X)\}$
$\leq d\epsilon_1 \cdot \max\{\|m_1\|_p, \|(m_2\|_q\} = \epsilon_2 \leq \epsilon$.
This, in view of [11, Theorem 4.1], implies that, if we consider m as a member of $M(\mathcal{R}, E\hat{\otimes}F)$, then $h = f \otimes g$ is m-integrable and

$$p \otimes q\left(\int f\, dm - \sum_{i,j} f(x_i)g(y_j) m_1(A_i) m_2(B_j)\right) \leq \epsilon_2.$$

Let

$$u_1 = \int f\, dm_1 - \sum_{i=1}^{k} f(x_i) m_1(A_i), \quad u_2 = \int g\, dm_2 - \sum_{j=1}^{r} g(y_j) m_2(B_j.$$

and

$$u = \int h\, dm - \sum_{i=1}^{k}\sum_{j=1}^{r} f(x_i) g(y_j) \cdot m_1(A_i) m_2(B_j).$$

Then
$-\int h\, dm + (\int f\, dm_1) \otimes (\int g\, dm_2) =$

$= -u + u_1 \otimes u_2 + u_1 \otimes \left(\sum_{j=1}^r g(y_j) m_2(B_j)\right) + \left[\sum_{i=1}^k f(x_i) m_1(A_i)\right] \otimes u_2.$
But $p \otimes q(u_1 \otimes u_2) \leq \epsilon_1^2 \leq \epsilon$ and $p \otimes q(u) \leq \epsilon$. Also

$$p \otimes q \left(u_1 \otimes \left(\sum_{j=1}^{r} g(y_j) m_2(B_j) \right) \right) \leq \epsilon_1 d \cdot \|m_2\|_q \leq \epsilon$$

and

$$p \otimes q \left(\left[\sum_{i=1}^{k} f(x_i) m_1(A_i) \right] \otimes u_2 \right) \leq \epsilon.$$

Thus

$$p \otimes q \left(\int h \, dm - \left[\int f \, dm_1 \right] \otimes \left[\int g \, dm_2 \right] \right) \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary and E, F are Hausdorff, it follows that

$$\int h \, dm = \left(\int f \, dm_1 \right) \otimes \left(\int g \, dm_2 \right)$$

which completes the proof.

3 The Case of τ-Additive Measures

Theorem 3.1 *If $m_1 \in M_\tau(\mathcal{R}_1, E)$ and $m_2 \in M_\tau(\mathcal{R}_2, F)$, then*

$$m = m_1 \otimes m_2 \in M_\tau(\mathcal{R}, E \hat{\otimes} F).$$

Proof : Consider on X, Y the zero-dimensional topologies $\tau_{\mathcal{R}_1}$, $\tau_{\mathcal{R}_2}$, respectively and on $X \times Y$ the product topology which coincides with $\tau_{\mathcal{R}}$. By [10, Theorem 10.4], there exists a linear homeomorphism

$$\omega : (C_b(X), \beta_o) \otimes (C_b(Y), \beta_o) \longrightarrow (C_b(X \times Y, \beta_o)$$

onto a dense subspace M of $C_b(X \times Y)$, where $\omega(f \otimes g) = f \odot g$, for $f \in C_b(X)$, $g \in C_b(Y)$. The map $f \mapsto \int f \, dm_1$, from $C_b(X)$ to E, is β_o-continuous by [11, Theorem 4.13]. The same is true for the map $g \mapsto \int g \, dm_2$, from $C_b(Y)$ to F.Given $p \in cs(E)$ and $q \in cs(F)$, there are $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$ such that

$$p \left(\int f \, dm_1 \right) \leq \|\phi_1 f\|, \quad q \left(\int g \, dm_2 \right) \leq \|\phi_2 g\|$$

for all $f \in C_b(X)$, $g \in C_b(Y)$. The bilinear map

$$T : (C_b(X), \beta_o) \times (C_b(Y), \beta_o) \to E \hat{\otimes} F, \quad T(f,g) = \left(\int f \, dm_1 \right) \otimes \left(\int g \, dm_2 \right)$$

is continuous and so the induced linear map

$$\psi : G = (C_b(X), \beta_o) \otimes (C_b(Y), \beta_o) \longrightarrow E \hat{\otimes} F$$

is continuous. Let

$$v = \psi \circ \omega^{-1} : M \longrightarrow E\hat{\otimes}F.$$

Since M is β_o-dense in $C_b(X \times Y)$, there exists a continuous linear extension

$$\bar{v} : (C_b(X \times Y), \beta_o) \longrightarrow E\hat{\otimes}F.$$

In view of [11, Theorem 4.13], there exists a unique $\mu \in M_\tau(\mathcal{R}, E\hat{\otimes}F)$ such that $\bar{v}(h) = \int h\,d\mu$ for all $h \in C_b(X \times Y)$. For $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$, taking as h the characteristic function of $A \times B$, we get that

$$\mu(A \times B) = \left(\int \chi_A\,dm_1\right) \otimes \left(\int \chi_B\,dm_2\right) = m_1(A)m_2(B).$$

Thus $\mu(\mathcal{R}) \subset E \otimes F)$ and $\mu = m$ by Theorem 2.5.

For R_o a separating algebra of subsets of a set Z , G a complete Hausdorff locally convex space and $u \in M_\tau(\mathcal{R}_o, G)$, we denote by L_μ the space of all $f \in \mathbb{K}^Z$ which are (VR)-integrable with respect to μ. On L_μ we consider the locally convex topology generated by the seminorms $N_{\mu,p}$, $p \in cs(G)$.

Theorem 3.2 *Let $m_1 \in M_\tau(\mathcal{R}_1, E)$, $m_2 \in M_\tau(\mathcal{R}_2, F)$ and $m = m_1 \otimes m_2$. Then the projective tensor product $L_{m_1} \otimes L_{m_2}$ is topologically isomorphic to a dense subspace of L_m.*

Proof : Consider the bilinear map

$$T : L_{m_1} \times L_{m_2} \longrightarrow L_m, \quad T(f, g) = f \odot g.$$

Let $p \in cs(E)$, $q \in cs(F)$, $f \in L_{m_1}$, $g \in L_{m_2}$ and $h = f \odot g$. Applying Theorem 2.6, we get that

$$\|h\|_{N_{m,p\otimes q}} = \|f\|_{N_{m_1,p}} \cdot \|g\|_{N_{m_2,q}}$$

and so T is continuous. Consequently the induced linear mat

$$\psi : L_{m_1} \otimes L_{m_2} \longrightarrow L_m$$

is continuous. The map ψ is one-to-one. In fact, assume that $\sum_{k=1}^n f_k \odot g_k = 0$, where $f_k \in L_{m_1}$, $g_k \in L_{m_2}$. We will show, by induction on n, that $\sum_{k=1}^n f_k \otimes g_k = 0$. This is clearly true if $n = 1$ or if each f_k is zero. Suppose that it is true for $n-1$ and that, say, $f_n \neq 0$. Then g_n is a linear combination of $g_1, \ldots, g_{n-1}$, i.e. $g_n = \sum_{k=1}^{n-1} \lambda_k g_k$, $\lambda_k \in \mathbb{K}$. Thus

$$0 = \sum_{k=1}^{n} f_k \odot g_k = \sum_{k=1}^{n-1} f_k \odot g_k + \sum_{k=1}^{n-1} \lambda_k (f_n \odot g_k) = \sum_{k=1}^{n-1} (f_k + \lambda_k f_n) \odot g_k.$$

By our induction hypothesis, we have

$$0 = \sum_{k=1}^{n-1} (f_k + \lambda_k f_n) \otimes g_k = \sum_{k=1}^{n-1} f_k \otimes g_k + f_n \otimes \left(\sum_{k=1}^{n-1} \lambda_k g_k\right) = \sum_{k=1}^{n} f_k \otimes g_k,$$

which shows that ψ is one-to-one.
Claim For $q_m = \|\cdot\|_{N_{m,p\otimes q}}$, $q_{m_1} = \|\cdot\|_{N_{m_1,p}}$, $q_{m_2} = \|\cdot\|_{N_{m_2,q}}$, we have

$$q_m(\psi(h)) = (q_{m_1} \otimes q_{m_2})(h).$$

Indeed, if $h = \sum_{k=1}^n f_k \otimes g_k$, then

$$q_m(\psi(h)) = q_m\left(\sum_{k=1}^n \psi(f_k \otimes g_k)\right) \leq \max_k q_m(f_k \odot g_k) = \max_k q_{m_1}(f_k) \cdot q_{m_2}(g_k),$$

which shows that $q_m(\psi(h)) \leq (q_{m_1} \otimes q_{m_2})(h)$. On the other hand, for an arbitrary $0 < t < 1$, there exists a representation $h = \sum_{k=1}^N f_k \otimes g_k$ of h such that the set $g_1, \dots, g_N$ is t-orthogonal with respect to the seminorm q_{m_2}. Let $u = \psi(h)$. For $x \in X$, let

$$u^x : Y \to \mathbb{K},\ u^x(y) = u(x,y) = \sum_{k=1}^N f_k(x) g_k(y).$$

Then

$$\sup_{y\in Y} |u(x,y)| \cdot N_{m,p\otimes q}(x,y) = N_{m_1,p}(x) \cdot \sup_{y\in Y} |u^x(y)| \cdot N_{m_2,q}(y) = N_{m_1,p}(x) \cdot q_{m_2}(u^x)$$
$$\geq t \cdot N_{m_1,p}(x) \cdot \max_k |f_k(x)| \cdot q_{m_2})(g_k).$$

Thus

$$q_m(u) \geq t \cdot \max_k [q_{m_1}(f_k) \cdot q_{m_2}(g_k)] \geq t \cdot (q_{m_1} \otimes q_{m_2})(h).$$

Since $0 < t < 1$ was arbitrary, we get that $q_m(u) \geq (q_{m_1} \otimes q_{m_2})(h)$ and the claim follows. Thus

$$\psi : L_{m_1} \otimes L_{m_2} \longrightarrow G = \psi(L_{m_1} \otimes L_{m_2})$$

is a topological isomorphism.
Finally, G is dense in L_m. Indeed, for $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$, we have that $\chi_{A\times B} = \psi(\chi_A \otimes \chi_B)$. Since each member of $\mathcal{R}$ is a finite union of sets of the form $A \times B$, with $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$, it follows that $S(\mathcal{R}) \subset G$ and hence G is dense in L_m since this is true for $S(\mathcal{R})$. This completes the proof.

4 A Fubini's Theorem

We will first define the integral of a vector-valued function with respect to a vector measure. Let $\mathcal{R}_o$ be a separating algebra of subsets of X, and $\mu \in M(\mathcal{R}_o, F)$. For $A \in \mathcal{R}_o$, let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, \dots, A_n; x_1, \dots, x_n\}$, where $\{A_1, \dots, A_n\}$ is an $\mathcal{R}_o$-partition of A and $x_k \in A_k$. For $f \in E^X$ and $\alpha = \{A_1, \dots, A_n; x_1, \dots, x_n\}$, we define $\psi_\alpha(f,\mu) = \sum_{k=1}^n f(x_k) \otimes \mu(A_k) \in E \otimes F$. If the limit $\lim_\alpha \psi_\alpha(f,\mu)$ exists in $E\hat{\otimes}F$, then we will say that f is μ-integrable over A and we will denote this limit by $\int_A f\,d\mu$. For $A = X$, we will write simply $\int f\,d\mu$. It is easy to see that if f is μ-integrable over X, then f is μ-integrable over every $A \in \mathcal{R}_o$ and

$$\int_A f\,d\mu = \int \chi_A f\,d\mu.$$

Using an argument analogous to the one used for scalar measures ([9], Theorem 2.1).

Theorem 4.1 *An $f \in E^X$ is μ-integrable with respect to some $\mu \in M(\mathcal{R}_o, F)$ iff, for each $p \in cs(E)$ and each $q \in cs(F)$, there exists an $\mathcal{R}_o$-partition $\{A_1, \ldots, A_n\}$ of X such that $p(f(x) - f(y)) \cdot m_q(A_k) \leq \epsilon$, for all k, if $x, y \in A_k$. Moreover, in this case we have that*

$$p \otimes q \left(\int f\, d\mu - \sum_{k=1}^{n} f(x_k) \otimes \mu(A_k) \right) \leq \epsilon,$$

where we denote also by $p \otimes q$ the unique continuous extension to all of $E \hat{\otimes} F$.

Assume next that $\mu \in M_\tau(\mathcal{R}_o, F)$. For $f \in E^X$, $q \in cs(F)$, $p \in cs(E)$, let

$$\|f\|_{N_{\mu,p,q}} = \sup_{x \in X} p(f(x)) \cdot N_{\mu,q}(x).$$

Let Z_μ be the space of all $f \in E^X$ with $\|f\|_{N_{\mu,p,q}} < \infty$ for all $p \in cs(E)$ and $q \in cs(F)$. Each $\|\cdot\|_{N_{\mu,p,q}}$ is a seminorm on Z_μ. Let $S(\mathcal{R}_o, E)$ be the space of all E-valued $\mathcal{R}_o$-simple functions. It is easy to see that, for $f = \sum_{k=1}^{n} \chi_{A_k} s_k$, we have that

$$\int f\, d\mu = \sum_{k=1}^{n} s_k \otimes \mu(A_k) \quad \text{and} \quad p \otimes q \left(\int f\, d\mu \right) \leq \|f\|_{N_{\mu,p,q}}$$

for all $p \in cs(E)$, $q \in cs(F)$. Let

$$\pi : S(\mathcal{R}_o, E) \longrightarrow E \hat{\otimes} F, \quad \pi(f) = \int f\, d\mu.$$

Then π is continuous if we consider on $S(\mathcal{R}_o, E)$ the topology induced by the topology of Z_μ. Thus there exists a continuous extension

$$\bar{\pi} : \overline{S(\mathcal{R}_o, E)} \longrightarrow E \hat{\otimes} F.$$

Definition 4.2 *A function $f \in E^X$ is said to be (VR)-integrable with respect to some $\mu \in M_\tau(\mathcal{R}_o, F)$ if it belongs to $D_\mu = \overline{S(\mathcal{R}_o, E)}$. In this case, $\bar{\pi}(f)$ is called the (VR)-integrable of f and will be denoted by $(VR) \int f\, d\mu$.*

Theorem 4.3 *If $f \in D_\mu$, then for all $p \in cs(E)$, $q \in cs(F)$, we have*

$$p \otimes q(\left((VR) \int f\, d\mu \right) \leq \|f\|_{N_{\mu,p,q}}.$$

Proof : There exists a net (g_δ) in $S(\mathcal{R}_o, E)$ converging to f. Then

$$(VR) \int f\, d\mu = \lim_{\delta} \int g_\delta d\mu, \quad \text{and} \quad \|g_\delta\|_{N_{\mu,p,q}} \to \|f\|_{N_{\mu,p,q}}.$$

Since

$$p \otimes q \left(\int g_\delta\, d\mu \right) \leq \|g_\delta\|_{N_{\mu,p,q}},$$

the Theorem follows.

In view of [11, Theorem 2.8], the closure of the set

$$\bigcup_{q \in cs(F)} \{x : N_{\mu,q}(x) > 0\}$$

is the smallest closed support set for μ.

Theorem 4.4 *Let $\mu \in M_\tau(\mathcal{R}_o, F)$ and $f \in E^X$. If f is μ-integrable, then f is also (VR)-integrable and*

$$\int f\,d\mu = (VR)\int f\,d\mu.$$

Proof : Assume that f is μ-integrable and let $\mathcal{D}$ be the directed set of all $\alpha = \{A_1 \dots, A_n; x_1, \dots, x_n\}$, where $\{A_1, \dots, A_n\}$ is an $\mathcal{R}_o$-partition of X and $x_k \in A_k$. Let $h_\alpha = \sum_{k=1}^n \chi_{A_k} f(x_k)$. Let $p \in cs(E)$, $q \in cs(F)$ and $\epsilon > 0$. In view of Theorem 4.1, there exist an $\mathcal{R}_o$-partition $\{B_1, \dots, B_N\}$ of X such that

$$\max_k \sup_{x,y \in B_k} p(f(x) - f(y)) \cdot m_q(B_k) \le \epsilon.$$

Let $x_k \in B_k$ and $\alpha_o = \{B_1, \dots, B_N; x_1, \dots, x_N\}$. Then , for

$$\alpha = \{A_1, \dots, A_n; y_1, \dots y_n\} \in \mathcal{D}, \quad \alpha \ge \alpha_o,$$

we have that

$$p \otimes q\left(\int f\,d\mu - \int h_\alpha\,d\mu\right) \le \epsilon.$$

It follows that $\|f\|_{N_{\mu,p,q}} < \infty$ and that

$$\int f\,d\mu = \lim_\alpha \int h_\alpha\,d\mu = (VR)\int f\,d\mu.$$

Hence the Theorem holds.

Lemma 4.5 *Let $m_1 \in M_\tau(\mathcal{R}_1, E)$, $m_2 \in M_\tau(\mathcal{R}_2, F)$ and $m = m_1 \otimes m_2$. Let also $f \in \mathbb{K}^{X \times Y}$ be (VR)-integrable with respect to m. For $y \in Y$, let $f^y = f(\cdot, y)$. If there exists a $q \in cs(F)$ such that $N_{m_2,q}(y) > 0$, then f^y is (VR)-integrable with respect to m_1.*

Proof : Every $h \in S(\mathcal{R})$ is of the form $h = \sum_{k=1}^n \lambda_k \chi_{A_k \times B_k}$, where $\lambda_k \in \mathbb{K}$, $A_k \in \mathcal{R}_1$, $B_k \in \mathcal{R}_2$. It is clear that $h^y \in S(\mathcal{R}_1)$. Suppose that $N_{m_2,q}(y) = d > 0$ and let $\epsilon > 0$. Given $p \in cs(E)$, there exists an $h \in S(\mathcal{R})$ such that $\|f - h\|_{N_{m,p\otimes q}} \le d\epsilon$. Now, for $x \in X$, we have

$$|f^y(x) - h^y(x)| \cdot N_{m_1,p}(x) = |f(x,y) - h(x,y)| \cdot N_{m,p\otimes q}(x,y)/d \le \epsilon$$

and so $\|f^y - h^y\|_{N_{m_1,p}} \le \epsilon$. This clearly proves that f^y is (VR)-integrable with respect to m_1.

Theorem 4.6 *(Fubini's Theorem). Let m_1, m_2, m be as in the preceding Lemma and let $f \in \mathbb{K}^{X \times Y}$ be (VR)-integrable with respect to m. Let $g : Y \to E$ be defined by $g(y) = (VR) \int f^y \, dm_1$ if*

$$y \in G = \{z \in Y : \exists q \in cs(F) \quad \text{with} \quad N_{m_2,q}(y) > 0\}$$

and arbitrarily if $y \notin G$. Then g is (VR)-integrable with respect to m_2 and

$$(VR) \int g \, dm_2 = (VR) \int f \, dm.$$

Proof : There exists a net (h_δ) in $S(\mathcal{R})$ such that $h_\delta \to f$ in L_m and

$$(VR) \int f \, dm = \lim_{\delta} \int h_\delta \, dm.$$

Define

$$g_\delta : Y \to E, \quad g_\delta(y) = \int h_\delta^y \, dm_1.$$

Then, for $p \in cs(E)$, $q \in cs(F)$ and $y \in Y$, we have

$$p(g_\delta(y) - g(y)) \cdot N_{m_2,q}(y) \le \|f - h_\delta\|_{N_{m,p \otimes q}}.$$

Indeed, if $N_{m_2,q}(y) \neq 0$, then

$$g_\delta(y) - g(y) = \int (f^y - h_\delta^y) \, dm_1$$

and so

$$p(g_\delta(y) - g(y)) \le \|f^y - h_\delta^y\|_{N_{m_1,p}},$$

which implies that

$$\begin{aligned} p(g_\delta(y) - g(y) \cdot N_{m_2,q}(y) &\le \sup_x |f(x,y) - h_\delta(x,y)| \cdot N_{m_1,p}(x) \cdot N_{m_2,q}(y) \\ &\le \|f - h_\delta\|_{N_{m,p \otimes q}}. \end{aligned}$$

This proves that g is (VR)-integrable with respect to m_2. Since, for $A \in \mathcal{R}_1$, $B \in \mathcal{R}_2$, $u = \chi_{A \times B}$ and $v(y) = \int u^y \, dm_1$,, we have that

$$\int h_\delta \, dm = \int g_\delta \, dm_2,$$

it follows that

$$\int h_\delta \, dm = \int g_\delta \, dm_2,$$

and so

$$(VR) \int f \, dm = \lim \int h_\delta \, dm = \lim \int g_\delta \, dm_2 = (VR) \int g \, dm_2,$$

which completes the proof.

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